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The Logical Quantization of Algebraic Groups

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In a previous paper we introduced a highly abstract framework within which the theory of manuals initiated by Foulis and Randall is to be developed. The framework enabled us in a subsequent paper to quantize the notion of a set. Following these lines, this paper is devoted to quantizing algebraic groups viewed from Grothendieck's functorial standpoint.

VISTA

In a previous paper (Nishimura, 1995b) we quantized the notion of a set with respect to a manual $\mathfrak{M}$ of Boolean locales. There, quantized sets were called empirical sets, which are, roughly speaking, assignments to each Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ of a Boolean valued set with respect to the complete Boolean algebra $\mathcal{P}(\mathbf{X})$ corresponding to $\mathbf{X}$. By way of example, the Booleanized sets of real numbers over Boolean locales $\mathbf{X}$ in $\mathfrak{M}$ lump together to form the empirical set of real numbers over $\mathfrak{M}$.

Set theory is the foundation of mathematics in the sense that every branch of mathematics, ranging from algebraic geometry to functional analysis, can be developed in principle within such a formal framework of axiomatic set theory as Zermelo–Fraenkel set theory with the axiom of choice. Now that the notion of a set is quantized, we are naturally inclined to consider quantizations of other mathematical structures, since "lower structures determine upper structures", as Engels put it. This paper is devoted to quantizing algebraic groups, though our exposition is intentionally oriented toward the general theory of (first) quantization. Our theory of quantized algebraic groups enables the lump of Booleanized real general linear groups of degree n for all the Boolean locales $\mathbf{X}$ in $\mathfrak{M}$ to get its due conceptual status.

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Linear groups such as GL_n and SL_n can be seen both as Lie groups and as algebraic groups. Lie groups are smooth manifolds endowed with a group structure definable by smooth functions, while affine algebraic groups are algebraic sets endowed with a group structure definable by polynomial functions. In spite of their vast technical discrepancies, the theory of Lie groups and that of algebraic groups enjoy immense similarity, for which the reader is referred, e.g., to Onishchik and Vinberg (1990). This paper is concerned with algebraic groups.

Algebraic sets can be approached from two distinct but complementary viewpoints. Geometrically, they are sets in affine spaces definable by families of polynomial functions. On the other hand, Grothendieck and others have espoused and expanded a functorial viewpoint that they are functors assigning sets to rings (more generally, to k-algebras for some fixed ring k). For a good introduction to the functorial treatment of algebraic geometry and algebraic groups in particular, the reader is referred to Demazure and Gabriel (1980). Waterhouse's (1979) monograph is also readable. Our treatment of algebraic groups in this paper is persistently functorial.

Interestingly enough, logical quantization should be preceded by Booleanization. Therefore, after fixing our terminology and notation of category theory in Section 0, we devote Section 1 to Booleanizing category theory. We suppose that Booleanization is now a common machinery in the arsenal of every working mathematician, for which the reader is referred, by way of example, to Nishimura (1984, 1992) and Takeuti (1978). Therefore, instead of producing the details of Booleanized proofs of Booleanized versions of well-known results of category theory in Section 1, we are content to produce Section 4, as an appendix, on what set theory we have in mind and how to interpret it. After discussing the relationship between two Booleanizations over possibly distinct complete Boolean algebras in Section 2, we discuss logical quantization of group-functors and some related concepts in Section 3. A category-theoretic viewpoint is conspicuous throughout the paper.

Conversance with our previous papers (Nishimura, 1993, 1995a,b) is highly helpful, but we are not necessarily faithful to our previous notation or terminology. A ring always means a commutative ring with unity in this paper, so that a homomorphism of rings is naturally required to preserve unities.

0. CATEGORY THEORY

The notion of a category is a generalization of that of a monoid on the one hand and a generalization of that of a poset on the other. Formally speaking, a category $\mathbf{C}$ is a 6-tuple (Ob $\mathbf{C}$, Mor $\mathbf{C}$, $d_{\mathbf{C}}$, $r_{\mathbf{C}}$, $id_{\mathbf{C}}$, $\circ_{\mathbf{C}}$), where:

(0.1) Ob **C** is a set whose elements are called objects.
(0.2) Mor **C** is a set whose elements are called morphisms.
(0.3) d_C and r_C are functions from Mor **C** to Ob **C**.
(0.4) id_C is a function from Ob **C** to Mor **C** such that $d_C(id_C(x)) = r_C(id_C(x)) = x$ for any $x \in$ Ob **C**.
(0.5) $\circ_C$ is a function from

$$\text{Mor } \mathbf{C} \times_{\text{Ob } \mathbf{C}} \text{Mor } \mathbf{C} = \{(g, f) \in \text{Mor } \mathbf{C} \times \text{Mor } \mathbf{C} \mid d_C(g) = r_C(f)\}$$

to Mor **C** (the value of $\circ_C$ at (g, f) is usually denoted by $g \circ_C f$) such that $d_C(g \circ_C f) = d_C(f)$ and $r_C(g \circ_C f) = r_C(g)$ for any $(g, f) \in \text{Mor } \mathbf{C} \times_{\text{Ob } \mathbf{C}} \text{Mor } \mathbf{C}$.
(0.6) $\circ_C$ is required to satisfy a certain kind of associative law, and $id_C(x)$ is required to play the role of two-sided identity for each $x \in$ Ob **C**.

We assume that the reader is well conversant with the fundamentals of category theory, for which the standard reference is, beyond all doubt, MacLane (1971). In particular, the reader should feel at home with such locutions as a functor, a natural transformation, the opposite category $\mathbf{C}^{op}$ of a category **C**, a limit, a colimit, etc.

To dodge the famous paradoxes of set theory or to paper them over, the usage of a *universe* U is a common practice in category theory, though the exact definition of a universe varies from one author to another in small details. For an exact definition of a universe, the reader is referred, e.g., to MacLane (1971, Chapter I, §6). In this paper we use two universes U, V with $U \in V$. A set of U (of V, resp.) is called *small*$_0$ (*small*$_1$, resp.). A category **C** is called *small*$_i$ if the set Mor **C** is small$_i$ ($i = 0, 1$). A category **C** is said to be *small*$_i$*-complete* (*small*$_i$*-cocomplete*, resp.) if all small$_i$ diagrams have limits (colimits, resp.) in **C** ($i = 0, 1$).

It is well known that such equationally definable algebraic systems as monoids, groups, and rings are definable within any category **C** with finite products (a terminal object should be regarded as a product of the empty family of objects) by using diagrams, for which the reader is referred to MacLane (1971, Chapter III, §6). We denote by **Mon(C)**, **Grp(C)** and **Rng(C)** the categories of monoids, groups, and rings in **C**, respectively. If **F**: **C** $\rightarrow$ **D** is a functor preserving finite limits, then it naturally induces functors $\mathbf{F}_{\mathbf{Mon}}$: **Mon(C)** $\rightarrow$ **Mon(D)**, $\mathbf{F}_{\mathbf{Grp}}$: **Grp(C)** $\rightarrow$ **Grp(D)**, and $\mathbf{F}_{\mathbf{Rng}}$: **Rng(C)** $\rightarrow$ **Rng(D)**. If R is an object of **Rng(C)**, then we can consider the categories $\mathbf{Alg}_R(\mathbf{C})$ and $\mathbf{Lie}_R(\mathbf{C})$ of R-algebras and Lie algebras over R within the category **C**. Let i be 0 or 1. We denote by $\mathbf{Ens}_i$ the category of small$_i$ sets and functions among them, which obviously has finite products. Then $\mathbf{Mon}(\mathbf{Ens}_i)$,

Grp(Ens$_i$), and **Rng(Ens$_i$)** denote the categories of small$_i$ monoids, small$_i$ groups, and small$_i$ rings in the usual sense, respectively. If R is a small$_i$ ring, then **Alg$_R$(Ens$_i$)** and **Lie$_R$(Ens$_i$)** denote the categories of small$_i$ R-algebras and small$_i$ Lie algebras over R in the usual sense, respectively.

We denote by **Bool** the category of complete Boolean algebras whose underlying sets are small$_0$. Its morphisms are all complete Boolean homomorphisms among such complete Boolean algebras. We denote by **BLoc** the opposite category **Bool**^{op} of the category **Bool**. The objects of **BLoc** are called *Boolean locales* and are denoted by **X**, **Y**, **Z**, ... Its morphisms are denoted by **f**, **g**, **h**, ... If a Boolean locale **X** is to be put down as an object of **Bool**, then it is denoted by $\mathcal{P}(\mathbf{X})$. Similarly, the morphism of **Bool** corresponding to $\mathbf{f} \in$ Mor **BLoc** is denoted by $\mathcal{P}(\mathbf{f})$. Given a Boolean locale **X** and $p \in \mathcal{P}(\mathbf{X})$, $\mathbf{X}_p$ denotes the Boolean locale such that $\mathcal{P}(\mathbf{X}_p) = \{q \in \mathcal{P}(\mathbf{X}) \mid q \leq p\}$. The canonical coproduct of two Boolean locales **X** and **Y** is denoted by $\mathbf{X} \oplus \mathbf{Y}$, for which $\mathcal{P}(\mathbf{X} \oplus \mathbf{Y}) = \mathcal{P}(\mathbf{X}) \times \mathcal{P}(\mathbf{Y})$.

An *orthogonal category* is a category endowed with a class of diagrams (called *orthogonal sum diagrams*) satisfying mild constraints. For the formal definition of an orthogonal category the reader is referred to Nishimura (1995a), but one important remark with regard to our usage of two kinds of smallness is in order. We will consider only small$_0$ orthogonal sum diagrams in this paper. This means, by way of example, that condition (2.11) of that paper is now to read as follows:

(2.11)$_0$ For any small$_0$ family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of objects in $\mathfrak{K}$ there exist an object **Y** in $\mathfrak{K}$ and a family $\{\mathbf{f}_\lambda\}_{\lambda \in \Lambda}$ of morphisms $\mathbf{f}_\lambda: \mathbf{X}_\lambda \to \mathbf{Y}$ in $\mathfrak{K}$ such that the diagram $\{\mathbf{X}_\lambda \xrightarrow{\mathbf{f}_\lambda} \mathbf{Y}\}_{\lambda \in \Lambda}$ lies in $\mathfrak{o}\mathfrak{F}_{\mathfrak{K}}$.

The due modifications for the other conditions of the definition of an orthogonal category which follow on our adherence to smallness$_0$ are safely entrusted to the reader. The motivating model of an orthogonal category is the opposite category of the category of small$_0$ complex Hilbert spaces and contractive linear transformations among them, in which the class of orthogonal sum diagrams is the categorical incarnation of the familiar notion of the orthogonal sum of a small$_0$ family of small$_0$ complex Hilbert spaces, and which was investigated in detail by Nishimura (1994) before the formal introduction of the notion of an orthogonal category. Another leading example of an orthogonal category is the category **BLoc**, in which the class of orthogonal sum diagrams is that of small$_0$ coproduct diagrams.

Orthogonal categories provide an abstract vehicle upon which the formal theory of manuals initiated by Foulis and Randall (1972; Randall and Foulis, 1973) is to be developed. In particular, the theory of manuals upon the orthogonal category **BLoc** studied in detail by Nishimura (1993) was the starting point of our long odyssey toward the formal theory of logical quantization. Formally, a *manual* in an orthogonal category **C** is a small_0 subcategory $\mathfrak{M}$ of the orthogonal category **C** satisfying certain mild constraints, under which the logic of $\mathfrak{M}$ is guaranteed to be an orthomodular poset. In this paper a *manual of Boolean locales* always means a completely coherent rich manual in the orthogonal category **BLoc**.

1. BOOLEANIZATION

Let **X** be an arbitrary Boolean locale, which shall be fixed throughout this section. We often denote $\mathcal{P}(\mathbf{X})$ by **B**. A family $\{p_\lambda\}_{\lambda\in\Lambda}$ of nonzero elements of **B** is called a *partition of unity* of **B** if it satisfies the following conditions:

(1.1) $\sup_{\lambda\in\Lambda}\mathbf{p}_\lambda = 1_\mathbf{B}$, where $1_\mathbf{B}$ is the unit of **B**.

(1.2) $p_\lambda \wedge p_{\lambda'} = 0_\mathbf{B}$ for any $\lambda, \lambda' \in \wedge$ with $\lambda \neq \lambda'$, where $0_\mathbf{B}$ is the zero of **B**.

A **B**-*valued set* is a pair $(\mathcal{U}, [\![\cdot=\cdot]\!]^{\mathcal{U}}_\mathbf{X})$ of a set and a function $[\![\cdot=\cdot]\!]^{\mathcal{U}}_\mathbf{X}$: $\mathcal{U} \times \mathcal{U} \to \mathbf{B}$ abiding by the following conditions:

(1.3) $[\![x = y]\!]^{\mathcal{U}}_\mathbf{X} = [\![y = x]\!]^{\mathcal{U}}_\mathbf{X}$

(1.4) $[\![x = y]\!]^{\mathcal{U}}_\mathbf{X} \wedge [\![y = z]\!]^{\mathcal{U}}_\mathbf{X} \leq [\![x = z]\!]^{\mathcal{U}}_\mathbf{X}$

for all $x, y, z \in \mathcal{U}$. We will often write $[\![x = y]\!]_\mathbf{X}$, $[\![x = y]\!]^{\mathcal{U}}$, or $[\![x = y]\!]$ for $[\![x = y]\!]^{\mathcal{U}}_\mathbf{X}$, unless confusion may arise.

Given a **B**-valued set $(\mathcal{U}, [\![\cdot=\cdot]\!])$, a function α: $\mathcal{U} \to \mathbf{B}$ is called a *singleton* if it satisfies the following conditions:

(1.5) $\alpha(x) \wedge [\![x = y]\!] \leq \alpha(y)$

(1.6) $\alpha(x) \wedge \alpha(y) \leq [\![x = y]\!]$

for all $x, y \in \mathcal{U}$. It is easy to see that any $x \in \mathcal{U}$ gives rise to a singleton $\{x\}$ assigning, to each $y \in \mathcal{U}$, $[\![x = y]\!] \in \mathbf{B}$. The **B**-valued set $(\mathcal{U}, [\![\cdot=\cdot]\!])$ is called an **X**-*set* if every singleton is of the form $\{x\}$ for a unique $x \in \mathcal{U}$.

Let $(\mathcal{U}, [\![\cdot=\cdot]\!])$ be an **X**-set. For each $x \in \mathcal{U}$ and each $p \in \mathbf{B}$ the unique element of $\mathcal{U}$ corresponding to the singleton $y \in \mathcal{U} \mapsto [\![x = y]\!] \wedge p$ is denoted by $x\lceil p$. Unless confusion may arise, the underlying set $\mathcal{U}$ of the **X**-set $(\mathcal{U}, [\![\cdot=\cdot]\!])$ is simply called an **X**-set without making the function $[\![\cdot=\cdot]\!]$: $\mathcal{U} \times \mathcal{U} \to \mathbf{B}$ explicit. A subset $\mathcal{V}$ of $\mathcal{U}$ can naturally be regarded as a **B**-valued set with respect to the restriction of the function $[\![\cdot=\cdot]\!]$: $\mathcal{U} \times \mathcal{U} \to \mathbf{B}$ to $\mathcal{V} \times \mathcal{V}$. If this **B**-valued set $\mathcal{V}$ happens to be an **X**-set, then it is called an **X**-*subset* of $\mathcal{U}$. For each $p \in \mathbf{B}$, the set $\{x \in \mathcal{U} \mid [\![x = x]\!] \leq p\}$ can naturally be regarded as an $\mathbf{X}_p$-set, denoted by $\mathcal{U}\lceil p$. The **X**-set $\mathcal{U}$ is said to be *small*$_i$ if its underlying set is small_i ($i = 0, 1$). An element $x \in \mathcal{U}$ is called *total* if

$[\![x = x]\!] = 1_{\mathbf{B}}$. A family $\{x_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathcal{U}$ is called an **X**-*basis* of the *X*-set $\mathcal{U}$ if it satisfies the following conditions:

(1.7) $[\![x_\lambda = x_\lambda]\!] = 1_{\mathbf{B}}$ for any $\lambda \in \Lambda$.
(1.8) $[\![x_\lambda = x_{\lambda'}]\!] = 0_{\mathbf{B}}$ for any $\lambda, \lambda' \in \Lambda$ with $\lambda \neq \lambda'$.
(1.9) $[\![y = y]\!] = \sup_{\lambda \in \Lambda}[\![y = x_\lambda]\!]$ for any element $y \in \mathcal{U}$.

The **X**-set $\mathcal{U}$ is said to be **X**-*finite* if there exists a partition $\{p_\xi\}_\Xi$ of unity of **B** such that $\mathcal{U}\lceil p_\xi$ is of a finite $\mathbf{X}_{p_\xi}$-basis for each $\xi \in \Xi$.

Given **X**-sets $\mathcal{U}$ and $\mathcal{V}$, an **X**-*function* from $\mathcal{U}$ to $\mathcal{V}$ is a function f from the set $\mathcal{U}$ to the set $\mathcal{V}$ abiding by the following conditions:

(1.10) $[\![x = y]\!]^{\mathcal{U}} \leq [\![f(x) = f(y)]\!]^{\mathcal{V}}$
(1.11) $[\![f(x) = f(x)]\!]^{\mathcal{V}} \leq [\![x = x]\!]^{\mathcal{U}}$

for all $x, y \in \mathcal{U}$. By substituting x for y in (1.10) and taking (1.11) into account, we have

(1.12) $[\![x = x]\!]^{\mathcal{U}} = [\![f(x) = f(x)]\!]^{\mathcal{V}}$

for all $x \in \mathcal{U}$. For each $p \in \mathbf{B}$, the **X**-function f naturally induces an $\mathbf{X}_p$-function from the $\mathbf{X}_p$-set $\mathcal{U}\lceil p$ to the $\mathbf{X}_p$-set $\mathcal{V}\lceil p$, denoted by $\mathbf{f}\lceil p$. The **X**-function f is said to be *small*$_i$ if $\mathcal{U}$ and $\mathcal{V}$ are small$_i$ ($i = 0, 1$).

It is well known that the category $\mathbf{BEns}_i(\mathbf{X})$ of small$_i$ **X**-sets and small$_i$ **X**-functions is a Boolean topos in which the axiom of choice holds ($i = 0, 1$). The canonical embedding of $\mathbf{BEns}_0(\mathbf{X})$ into $\mathbf{BEns}_1(\mathbf{X})$ is a logical morphism. The category $\mathbf{BEns}_1(\mathbf{X})$ is small$_1$-complete and small$_1$-cocomplete, and the category $\mathbf{BEns}_0(\mathbf{X})$ is closed under small$_0$ limits and small$_0$ colimits in $\mathbf{BEns}_1(\mathbf{X})$. Given **X**-sets $\mathcal{U}$ and $\mathcal{V}$, the underlying set of their canonical product $\mathcal{U} \times_{\mathbf{X}} \mathcal{V}$ is

$$\{(x, y) \in \mathcal{U} \times \mathcal{V} \mid [\![x = x]\!]^{\mathcal{U}} = [\![y = y]\!]^{\mathcal{V}}\}$$

The category $\mathbf{Rng}(\mathbf{BEns}_0(\mathbf{X}))$ is denoted by $\mathbf{BRng}(\mathbf{X})$. Given an object $\mathcal{R}$ of $\mathbf{BRng}(\mathbf{X})$, the category $\mathbf{Alg}_{\mathcal{R}}(\mathbf{BEns}_0(\mathbf{X}))$ is denoted by $\mathbf{BAlg}_{\mathcal{R}}(\mathbf{X})$.

An **X**-category $\mathcal{C}$ is a 6-tuple (Ob $\mathcal{C}$, Mor $\mathcal{C}$, $\mathrm{d}_{\mathcal{C}}$, $\mathrm{r}_{\mathcal{C}}$, $\mathrm{id}_{\mathcal{C}}$, $\circ_{\mathcal{C}}$), where:

(1.13) Ob $\mathcal{C}$ and Mor $\mathcal{C}$ are **X**-sets.
(1.14) $\mathrm{d}_{\mathcal{C}}$ and $\mathrm{r}_{\mathcal{C}}$ are **X**-functions from Mor $\mathcal{C}$ to Ob $\mathcal{C}$.
(1.15) $\mathrm{id}_{\mathcal{C}}$ is an **X**-function from Ob $\mathcal{C}$ to Mor $\mathcal{C}$ such that $[\![x = y]\!]^{\mathrm{Ob}\,\mathcal{C}} = [\![\mathrm{id}_{\mathcal{C}}(x) = \mathrm{id}_{\mathcal{C}}(y)]\!]^{\mathrm{Mor}\,\mathcal{C}}$ for all $x, y \in \mathrm{Ob}\,\mathcal{C}$.
(1.16) $\circ_{\mathcal{C}}$ is an **X**-function from

$$\mathrm{Mor}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C} = \{(g, f) \in \mathrm{Mor}\,\mathcal{C} \times \mathrm{Mor}\,\mathcal{C} \mid \mathrm{d}_{\mathcal{C}}(g) = r_{\mathcal{C}}(f)\}$$

to Mor $\mathcal{C}$, where we note that $\mathrm{Mor}\,\mathcal{C} \times_{\mathrm{Ob}\,\mathcal{C}} \mathrm{Mor}\,\mathcal{C}$ is an **X**-subset of $\mathrm{Mor}\,\mathcal{C} \times_{\mathbf{X}} \mathrm{Mor}\,\mathcal{C}$.

(1.17) The 6-tuple (Ob $\mathscr{C}$, Mor $\mathscr{C}$, $d_{\mathscr{C}}$, $r_{\mathscr{C}}$, $id_{\mathscr{C}}$, $\circ_{\mathscr{C}}$) is a category in the usual sense.

Unless confusion may arise, the subscript $\mathscr{C}$ in $d_{\mathscr{C}}$, $r_{\mathscr{C}}$, $id_{\mathscr{C}}$, and $\circ_{\mathscr{C}}$ is usually omitted. We usually denote the value of (g, f) under $\circ$ by $g \circ f$ rather than $\circ(g, f)$. Of course, elements of Ob $\mathscr{C}$ are called *objects* of $\mathscr{C}$ and elements of Mor $\mathscr{C}$ are called *morphisms* of $\mathscr{C}$. Total elements of Ob $\mathscr{C}$ are called *total objects* of $\mathscr{C}$, and total elements of Mor $\mathscr{C}$ are called *total morphisms* of $\mathscr{C}$. Given $x, y \in$ Ob $\mathscr{C}$, the totality of morphisms f: $x\lceil p \to y\lceil p$ for all $p \in \mathbf{B}$ with $p \leq [\![x = x]\!] \wedge [\![y = y]\!]$ is denoted by $\mathscr{C}_{\mathbf{X}}(x, y)$. The **X**-category $\mathscr{C}$ is said to be *small$_i$* if the **X**-set Ob $\mathscr{C}$ is small$_i$ ($i = 0, 1$). The **X**-category $\mathscr{C}$ is said be **X**-*finite* if the **X**-set Mor $\mathscr{C}$ is **X**-finite. The *opposite* **X**-category of the **X**-category $\mathscr{C}$ is defined as expected, and is denoted by $\mathscr{C}^{op}$. Given $p \in \mathbf{B}$, the full subcategory of the category $\mathscr{C}$ whose objects are all $x \in$ Ob $\mathscr{C}$ with $[\![x = x]\!] \leq p$ can naturally be rated as an $\mathbf{X}_p$-category and is denoted by $\mathscr{C}\lceil p$. Given $p \in \mathbf{B}$, the full subcategory of the category $\mathscr{C}$ whose objects are all $x \in$ Ob $\mathscr{C}$ with $[\![x = x]\!] = p$ is called the *p-slice* of the **X**-category $\mathscr{C}$ and is denoted by $\mathscr{C}[p]$. An **X**-category $\mathscr{D}$ = (Ob $\mathscr{D}$, Mor $\mathscr{D}$, $d_{\mathscr{D}}$, $r_{\mathscr{D}}$, $id_{\mathscr{D}}$, $\circ_{\mathscr{D}}$) is said to be an **X**-*subcategory* of the **X**-category $\mathscr{C}$ = (Ob $\mathscr{C}$, Mor $\mathscr{C}$, $d_{\mathscr{C}}$, $r_{\mathscr{C}}$, $id_{\mathscr{C}}$, $\circ_{\mathscr{C}}$) if the underlying category of $\mathscr{D}$ is a subcategory of that of $\mathscr{C}$, and if the **X**-sets Ob $\mathscr{D}$ and Mor $\mathscr{D}$ are **X**-subsets of Ob $\mathscr{C}$ and Mor $\mathscr{C}$, respectively.

Now an example of an **X**-category will be instructive.

Example 1.1. Let i be 0 or 1. A *small$_i$ partial* **X**-*set* is an ordered pair $(p, \mathscr{U})$ of an element p of **B** and a small$_i$ $\mathbf{X}_p$-set $\mathscr{U}$. A *small$_i$ partial* **X**-*function* is an ordered pair $(p, \mathscr{f})$ of an element p of **B** and a small$_i$ $\mathbf{X}_p$-function $\mathscr{f}$. The totality of small$_i$ partial **X**-sets can be regarded as an **X**-set by defining $[\![(p, \mathscr{U}) = (q, \mathscr{V})]\!]$ to be

$$(1.18)\quad [\![(p, \mathscr{U}) = (q, \mathscr{V})]\!] = \sup\{r \in \mathbf{B} \mid r \leq p \wedge q \text{ and } \mathscr{U}\lceil r = \mathscr{V}\lceil r\}$$

Two small$_i$ partial **X**-sets $(p, \mathscr{U})$ and $(p, \mathscr{V})$ whose first components happen to be the same p are identified provided that $[\![(p, \mathscr{U}) = (p, \mathscr{V})]\!] = p$. The totality of small$_i$ partial **X**-functions can similarly be rated as an **X**-set by defining $[\![(p, \mathscr{f}) = (q, \mathscr{g})]\!]$ to be

$$(1.19)\quad [\![(p, \mathscr{f}) = (q, \mathscr{g})]\!] = \sup\{r \in \mathbf{B} \mid r \leq p \wedge q \text{ and } \mathscr{f}\lceil r = \mathscr{g}\lceil r\}.$$

Two small$_i$ partial **X**-functions $(p, \mathscr{f})$ and $(p, \mathscr{g})$ whose first components happen to be the same p are identified provided that $[\![(p, \mathscr{f}) = (p, \mathscr{g})]\!] = p$. The totality of small$_i$ partial **X**-sets and small$_i$ partial **X**-functions naturally forms an **X**-category $\mathscr{BEns}_i(\mathbf{X})$. It is easy to see that for each $p \in \mathbf{B}$, the category $\mathscr{BEns}_i(\mathbf{X})[p]$ is equivalent to the category $\mathbf{BEns}_i(\mathbf{X}_p)$. ■

An **X**-*functor* from an **X**-category $\mathscr{C}$ to an **X**-category $\mathscr{D}$ is a functor $\mathscr{F}$ from the underlying category $\mathscr{C}$ to the underlying category $\mathscr{D}$ abiding by the following condition:

(1.20) The assignment $f \in \mathrm{Mor}\,\mathscr{C} \mapsto \mathscr{F}(f) \in \mathrm{Mor}\,\mathscr{D}$ is an **X**-function.

Proposition 1.2. An **X**-functor $\mathscr{F}$ from an **X**-category $\mathscr{C}$ to an **X**-category $\mathscr{D}$ satisfies the following condition:

(1.21) The assignment $x \in \mathrm{Ob}\,\mathscr{C} \mapsto \mathscr{F}(x) \in \mathrm{Ob}\,\mathscr{D}$ is an **X**-function.

Proof. For any $x, y \in \mathrm{Ob}\,\mathscr{C}$, we have

$$\begin{aligned}[\![x = y]\!]^{\mathrm{Ob}\,\mathscr{C}} &= [\![\mathrm{id}(x) = \mathrm{id}(y)]\!]^{\mathrm{Mor}\,\mathscr{C}} \\ &\leq [\![\mathscr{F}(\mathrm{id}(x)) = \mathscr{F}(\mathrm{id}(y))]\!]^{\mathrm{Mor}\,\mathscr{D}} \\ &= [\![\mathrm{id}(\mathscr{F}(x)) = \mathrm{id}(\mathscr{F}(y))]\!]^{\mathrm{Mor}\,\mathscr{D}} \\ &= [\![\mathscr{F}(x) = \mathscr{F}(y)]\!]^{\mathrm{Ob}\,\mathscr{D}} \quad \blacksquare\end{aligned}$$

An **X**-functor $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ naturally gives rise to a functor $\mathscr{F}[p]: \mathscr{C}[p] \to \mathscr{D}[p]$ and an $\mathbf{X}_p$-functor $\mathscr{F}\lceil p: \mathscr{C}\lceil p \to \mathscr{D}\lceil p$ for every $p \in \mathbf{B}$. A *contravariant* **X**-*functor* from an **X**-category $\mathscr{C}$ to an **X**-category $\mathscr{D}$ is an **X**-functor from the **X**-category $\mathscr{C}^{\mathrm{op}}$ to the **X**-category $\mathscr{D}$.

Given an **X**-category $\mathscr{J}$ and an **X**-category $\mathscr{D}$, an **X**-*diagram* in $\mathscr{C}$ of type $\mathscr{J}$ is an **X**-*functor* $\mathscr{F}$ from $\mathscr{J}$ to $\mathscr{C}$. More generally, a *partial* **X**-*diagram* in an **X**-category $\mathscr{C}$ is an ordered pair of $(p, \mathscr{F})$ of $p \in \mathbf{B}$ and an $\mathbf{X}_p$-diagram $\mathscr{F}$ in $\mathscr{C}\lceil p$, in which the type of the $\mathbf{X}_p$-diagram $\mathscr{F}$ is called the *type* of the partial **X**-diagram. An **X**-diagram in an **X**-category $\mathscr{C}$ is said to be **X**-*finite, small*$_0$ or *small*$_1$ if its type is so accordingly. Similarly, a partial **X**-diagram $(p, \mathscr{F})$ is said to be **X**-*finite, small*$_0$ or *small*$_1$ if the type of the $\mathbf{X}_p$-diagram $\mathscr{F}$ is $\mathbf{X}_p$-finite, small$_0$ or small$_1$ accordingly.

Example 1.3. Let $\mathscr{C}$ and $\mathscr{D}$ be **X**-categories and x an object of $\mathscr{D}$. Let $p = [\![x = x]\!]$. Then the assignments $y \in \mathrm{Ob}\,\mathscr{C}\rceil p \mapsto x\lceil[\![y = y]\!] \in \mathrm{Ob}\,\mathscr{D}$ and $f \in \mathrm{Mor}\,\mathscr{C}\rceil p \mapsto \mathrm{id}(x)\lceil[\![y = y]\!] \in \mathrm{Mor}\,\mathscr{D}$ constitute an $\mathbf{X}_p$-functor to be denoted by $\langle x\rangle_{\mathscr{C}}$. ■

Given **X**-categories $\mathscr{C}$ and $\mathscr{D}$, a *natural* **X**-*transformation* from an **X**-functor $\mathscr{F}: \mathscr{C} \to \mathscr{D}$ to an **X**-functor $\mathscr{G}: \mathscr{C} \to \mathscr{D}$ is a natural transformation σ from the functor $\mathscr{F}$ to the functor $\mathscr{G}$ abiding by the following condition:

(1.22) The assignment $x \in \mathrm{Ob}\,\mathscr{C} \mapsto \sigma(x) \in \mathrm{Mor}\,\mathscr{D}$ is an **X**-function.

A natural **X**-transformation $\sigma: \mathscr{F} \to \mathscr{G}$ is called a *natural* **X**-*isomorphism* provided that $\sigma(x)$ is an isomorphism for each $x \in \mathrm{Ob}\,\mathscr{C}$, in which $\mathscr{F}$ and $\mathscr{G}$ are said to be **X**-*equivalent*. Given two **X**-categories $\mathscr{C}$ and $\mathscr{D}$, if there exist functors $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}: \mathcal{D} \to \mathcal{C}$ such that $\mathcal{G} \circ \mathcal{F}$ and $\mathcal{F} \circ \mathcal{G}$ are **X**-equivalent to the identity **X**-functors $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$, respectively, then $\mathcal{C}$ and $\mathcal{D}$ are said to be **X**-*equivalent.*

Example 1.4. Let $\mathcal{C}$ and $\mathcal{D}$ be **X**-categories and $f: x \to y$ a morphism in $\mathcal{D}$. Let $p = [\![f = f]\!]$. Then f naturally induces a natural $\mathbf{X}_p$-transformation $\langle f \rangle_{\mathcal{C}}: \langle x \rangle_{\mathcal{C}} \to \langle y \rangle_{\mathcal{C}}$ assigning, to each $z \in \mathrm{Ob}\,\mathcal{C}$, $\langle f \rangle_{\mathcal{C}}(z) = f\lceil [\![z = z]\!]$. ■

Let $\mathcal{C}$ be an **X**-category and $(p, \mathcal{F})$ a partial **X**-diagram in $\mathcal{C}$ of type $\mathcal{J}$. An **X**-*cone* from $(p, \mathcal{F})$ is an ordered pair (x, σ) of an object x in $\mathcal{C}$ with $[\![x = x]\!] = p$ and a natural $\mathbf{X}_p$-transformation σ from the $\mathbf{X}_p$-functor $\mathcal{F}$ to the $\mathbf{X}_p$-functor $\langle x \rangle_{\mathcal{J}}$. An **X**-cone (x, σ) from the partial **X**-diagram $(p, \mathcal{F})$ is called an **X**-*colimit* of $(p, \mathcal{F})$ if it is universal in the sense that for any **X**-cone (y, ρ) from $(p, \mathcal{F})$ there exists a unique morphism f from x to y such that $\rho = \langle f \rangle_{\mathcal{J}} \circ \sigma$. The notion of an **X**-*cone* to $(p, \mathcal{F})$ and that of an **X**-*limit* of $(p, \mathcal{F})$ can be defined similarly.

An **X**-category $\mathcal{C}$ is said to be **X**-*finitely* **X**-*cocomplete* if any **X**-finite partial **X**-diagram in $\mathcal{C}$ has an **X**-colimit in $\mathcal{C}$. The notion of being *small*$_0$-**X**-*cocomplete* and that of being *small*$_1$-**X**-*cocomplete* are defined similarly. An **X**-category $\mathcal{C}$ is said to be **X**-*finitely* **X**-*complete* if any **X**-finite partial **X**-diagram in $\mathcal{C}$ has an **X**-limit in $\mathcal{C}$. The notion of being *small*$_0$-**X**-*complete* and that of being *small*$_1$-**X**-*complete* are defined similarly.

Example 1.5. The **X**-category $\mathcal{BC}ns_i(X)$ is small$_i$-X-complete and small$_i$-X-cocomplete ($i = 0, 1$). ■

With Theorem 7.4.2 of Schubert (1972) and its Booleanization in mind, it is easy to see the following result.

Proposition 1.6. An **X**-category $\mathcal{C}$ is **X**-finitely **X**-complete (**X**-finitely **X**-cocomplete, resp.) iff the category $\mathcal{C}[p]$ is finitely complete (finitely cocomplete, resp.) for every $p \in \mathbf{B}$. ■

An **X**-category $\mathcal{C}$ is called an **X**-*topos* if $\mathcal{C}[p]$ is a topos for every $p \in \mathbf{B}$. By way of example, $\mathcal{BC}ns_0(\mathbf{X})$ and $\mathcal{BC}ns_1(\mathbf{X})$ are **X**-toposes.

An **X**-functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is said to *preserve* **X**-*finite* **X**-*limits* provided that, whenever (x, σ) is an **X**-limit of an **X**-finite partial **X**-diagram $(p, \mathcal{G})$ in $\mathcal{C}$, then $(\mathcal{F}(x), \mathcal{F}\sigma)$ is an **X**-limit of the partial **X**-diagram $(p, (\mathcal{F}\lceil p) \circ \mathcal{G})$ in $\mathcal{D}$. The notion of *preserving small*$_0$ **X**-*limits* and that of *preserving small*$_1$ **X**-*limits* are defined similarly. Dually, an **X**-functor $\mathcal{F}: \mathcal{C} \to \mathcal{D}$ is said to *preserve* **X**-*finite* **X**-*colimits* provided that, whenever (x, σ) is an **X**-colimit of an **X**-finite partial **X**-diagram $(p, \mathcal{G})$ in $\mathcal{C}$, then $(\mathcal{F}(x), \mathcal{F}\sigma)$ is an **X**-colimit of the partial **X**-diagram $(p, (\mathcal{F}\lceil p) \circ \mathcal{G})$ in $\mathcal{D}$. The notion of *preserving small*$_0$ **X**-*colimits* and that of *preserving small*$_1$ **X**-*colimits* are defined similarly. As in Proposition 1.6, it is easy to see the following result.

Proposition 1.7. An **X**-functor $\mathcal{F}$: $\mathcal{C} \to \mathcal{D}$ preserves **X**-finite **X**-limits (**X**-finite **X**-colimits, resp.) iff the functor $\mathcal{F}[p]$: $\mathcal{C}[p] \to \mathcal{D}[p]$ preserves finite limits (finite colimits, resp.) for every $p \in \mathbf{B}$. ∎

Let $\mathcal{C}$ be an **X**-finitely **X**-complete **X**-category. Then the lump of **Grp**($\mathcal{C}[p]$)'s for all $p \in \mathbf{B}$ constitute an **X**-category denoted by $\mathcal{BG}rp(\mathcal{C})$ and called the **X**-*category of groups in* $\mathcal{C}$. Total objects of $\mathcal{BR}ng(\mathcal{C})$ are called *total groups* in $\mathcal{C}$. Similarly, the lump of **Rng**($\mathcal{C}[p]$)'s for all $p \in \mathbf{B}$ constitute an **X**-category denoted by $\mathcal{BR}ng(\mathcal{C})$ and called the **X**-*category of rings in* $\mathcal{C}$. Total objects of $\mathcal{BR}ng(\mathcal{C})$ are called *total rings* in $\mathcal{C}$. Given a total ring $\mathcal{R}$ in $\mathcal{C}$, the lump of $\mathbf{Alg}_{\mathcal{R}\lceil p}(\mathcal{C}[p])$'s for all $p \in \mathbf{B}$ constitute an **X**-category denoted by $\mathcal{BA}lg_{\mathcal{R}}(\mathcal{C})$ and called the **X**-*category of algebras over* $\mathcal{R}$ *in* $\mathcal{C}$. Similarly, the lump of $\mathbf{Lie}_{\mathcal{R}\lceil p}(\mathcal{C}[p])$'s for all $p \in \mathbf{B}$ constitute an **X**-category denoted by $\mathcal{BL}ie_{\mathcal{R}}(\mathcal{C})$ and called the **X**-*category of Lie algebras over* $\mathcal{R}$ *in* $\mathcal{C}$.

Example 1.8. The **X**-category $\mathcal{BR}ng(\mathcal{BE}ns_0(\mathbf{X}))$ is denoted by $\mathcal{BR}ng(\mathbf{X})$. For each $p \in \mathbf{B}$, the category $\mathcal{BR}ng(\mathbf{X})[p]$ is equivalent to the category $\mathbf{BRng}(\mathbf{X}_p)$. Let $\mathcal{R}$ be a total object of $\mathcal{BR}ng(\mathbf{X})$. The **X**-category $\mathcal{BA}lg_{\mathcal{R}}(\mathcal{BE}ns_0(\mathbf{X}))$ is denoted by $\mathcal{BA}lg_{\mathcal{R}}(\mathbf{X})$. For each $p \in \mathbf{B}$, the category $\mathcal{BA}lg_{\mathcal{R}}(\mathbf{X})[p]$ is equivalent to the category $\mathbf{BAlg}_{\mathcal{R}\lceil p}(X_p)$. ∎

Let $\mathcal{F}$: $\mathcal{C} \to \mathcal{D}$ be an **X**-functor of **X**-finitely **X**-complete **X**-categories preserving **X**-finite **X**-limits. Then it naturally induces **X**-functors $\mathcal{F}_{\mathcal{BG}rp}$: $\mathcal{BG}rp(\mathcal{C}) \to \mathcal{BG}rp(\mathcal{D})$ and $\mathcal{F}_{\mathcal{BR}ng}$: $\mathcal{BR}ng(\mathcal{C}) \to \mathcal{BR}ng(\mathcal{D})$.

Example 1.9. Let $\mathcal{C}$ be a small_1 **X**-category and $x \in \mathrm{Ob}\,\mathcal{C}$. Then the assignment $y \in \mathrm{Ob}(\mathcal{C}\lceil [\![x = x]\!]) \mapsto ([\![y = y]\!], \mathcal{C}_{\mathbf{X}}(x, y))$ with $\mathcal{C}_{\mathbf{X}}(x, y)$ being regarded as an $\mathbf{X}_{[\![y=y]\!]}$-set naturally induces an $\mathbf{X}_{[\![x=x]\!]}$-functor $\mathcal{C}_{\mathbf{X}}(x, \cdot)$ from $\mathcal{C}\lceil \|x = x\|$ to $\mathcal{BE}ns_1(\mathbf{X})\lceil \|x = x\|$. ∎

Example 1.10. Let $\mathcal{R}$, $\mathcal{P}$ be total objects in $\mathcal{BR}ng(\mathbf{X})$. Let f: $\mathcal{R} \to \mathcal{P}$ be a morphism in $\mathcal{BR}ng(\mathbf{X})$, so that $\mathcal{P}\lceil p$ can be rated as an algebra over $\mathcal{R}\lceil p$ via $f\lceil p$ in the Boolean topos $\mathbf{BEns}_0(\mathbf{X}_p)$ for every $p \in \mathbf{B}$. The lump of base extension functors

$$-\otimes_{f\lceil p}(\mathcal{P}\lceil p) \quad : \mathbf{BAlg}_{\mathcal{R}\lceil p}(\mathbf{X}_p) \to \mathbf{BAlg}_{\mathcal{P}\lceil p}(\mathbf{X}_p) \qquad \text{for all} \quad p \in \mathbf{B}$$

constitute an **X**-functor from $\mathcal{BA}lg_{\mathcal{R}}(\mathbf{X})$ to $\mathcal{BA}lg_{\mathcal{P}}(\mathbf{X})$, denoted by $-\otimes_f \mathcal{P}$ and called the *base extension* **X**-*functor of* f. ∎

Example 1.11. Let $\mathcal{C}$ be a small_1 **X**-category and $\mathcal{D}$ an **X**-category. The totality of ordered pairs $(p, \mathcal{F})$ with $p \in \mathbf{B}$ and $\mathcal{F}$ being an $\mathbf{X}_p$-functor from the $\mathbf{X}_p$-category $\mathcal{C}\lceil p$ to the $\mathbf{X}_p$-category $\mathcal{D}\lceil p$ can naturally be put down as an **X**-subset of Ob $\mathcal{BE}ns_1(\mathbf{X})$. The totality of ordered pairs (p, σ) with $p \in$

B and σ being a natural $\mathbf{X}_p$-transformation between $\mathbf{X}_p$-functors from the $\mathbf{X}_p$-category $\mathcal{C}\lceil p$ to the $\mathbf{X}_p$-category $\mathcal{D}\lceil p$ can naturally be put down as an **X**-subset of Mor $\mathcal{BE}ns_1(\mathbf{X})$. They constitute an **X**-subcategory of $\mathcal{BE}ns_1(\mathbf{X})$, denoted by $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{D})$. Its total objects are to be identified with **X**-functors from $\mathcal{C}$ to $\mathcal{D}$. ■

Examples 1.10 and 1.11 are combined to yield the following.

Example 1.12. Given a small_1 **X**-category $\mathcal{C}$, the assignment

$$x \in \text{Ob}\,\mathcal{C} \mapsto ([\![x = x]\!], \mathcal{C}_{\mathbf{X}}(\mathbf{x}, \cdot)) \in \text{Ob}\,\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X}))$$

naturally induces a contravariant **X**-functor **y** from $\mathcal{C}$ to $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X}))$, called the *Yoneda embedding*. An object of $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X}))$ is said to be *representable* if it is isomorphic to $\mathbf{y}(x)$ for some object $x \in \text{Ob}\,\mathcal{C}$, in which the object of $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X}))$ is said to *be represented by* x. ■

The following is only a Booleanization of Schubert's (1972) Theorems 7.5.2 and 8.5.1.

Theorem 1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be **X**-categories. If $\mathcal{D}$ is small_0-**X**-complete (small_1-**X**-complete, **X**-finitely **X**-complete, small_0-**X**-cocomplete, small_1-**X**-cocomplete, **X**-finitely **X**-complete, resp.), then so is the **X**-category $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{D})$. ■

Example 1.14. Let $\mathcal{R}$ be a total object of $\mathcal{BR}ng(\mathbf{X})$. The **X**-category $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{BA}lg_{\mathcal{R}}(\mathbf{X}), \mathcal{BE}ns_1(\mathbf{X}))$ is denoted by $\mathcal{BF}unct_{\mathcal{R}}(\mathbf{X})$, which is small_1-**X**-complete by Example 1.5 and Theorem 1.13. Therefore we can speak of the **X**-categories $\mathcal{BG}rp\,(\mathcal{BF}unct_{\mathcal{R}}(\mathbf{X}))$ and $\mathcal{BR}ng\,(\mathcal{BF}unct_{\mathcal{R}}(\mathbf{X}))$, which we denote by $\mathcal{BGF}unct_{\mathcal{R}}(\mathbf{X})$ and $\mathcal{BRF}unct_{\mathcal{R}}(\mathbf{X})$ respectively. The identity **X**-functor on $\mathcal{BA}lg_{\mathcal{R}}(\mathbf{X})$ can naturally be rated as an object of $\mathcal{BRF}unct_{\mathcal{R}}(\mathbf{X})$, which is denoted by $\mathcal{O}_{\mathbf{X},\mathcal{R}}$. If $\mathcal{O}_{\mathbf{X},\mathcal{R}}$ is regarded as an object of $\mathcal{BF}unct_{\mathcal{R}}(\mathbf{X})$, then it is representable, and its representing object is denoted by $\mathcal{R}[\mathbf{X}]$. The **X**-category $\mathcal{BL}ie_{\mathcal{O}_{\mathbf{X},\mathcal{R}}}(\mathcal{BF}unct_{\mathcal{R}}(\mathbf{X}))$ is denoted by $\mathcal{BLF}unct_{\mathcal{R}}(\mathbf{X})$. ■

The following is only a Booleanization of the well-known Yoneda lemma, for which the reader is referred, e.g., to MacLane (1971, Chapter III, §2).

Theorem 1.15. Let $\mathcal{C}$ be a small_1 **X**-category, x a total object of $\mathcal{C}$, and $\mathcal{F}$ an **X**-functor from $\mathcal{C}$ to $\mathcal{BE}ns_1(\mathbf{X})$. Then the assignment to each natural transformation σ: $\mathcal{C}_{\mathbf{X}}(x, \cdot) \to \mathcal{F}$ of $\sigma(x)(\text{id}(x))$ gives rise to a bijective correspondence between the natural **X**-transformations from $\mathcal{C}_{\mathbf{X}}(x, \cdot)$ to $\mathcal{F}$ and the elements of the underlying set of the **X**-set $\mathcal{F}(x)$. ■

2. RELATIONS BETWEEN TWO BOOLEANIZATIONS

Let $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ be an arbitrary morphism in **Bloc**, which shall be fixed throughout this section. As explained in Nishimura (1993, §2), two functors $\mathbf{f}^*: \mathbf{BEns}_1(\mathbf{Y}) \to \mathbf{BEns}_1(\mathbf{X})$ and $\mathbf{f}_*: \mathbf{BEns}_1(\mathbf{X}) \to \mathbf{BEns}_1(\mathbf{Y})$ are canonically constructed from the functor $\mathbf{f}$, for which $\mathbf{f}^* \dashv \mathbf{f}_*$ and $\mathbf{f}^*$ is left exact. For each $p \in \mathcal{P}(\mathbf{Y})$, $\mathbf{f}$ naturally induces a morphism $\mathbf{f}_p: \mathbf{X}_{\mathcal{P}(\mathbf{f})(p)} \to \mathbf{Y}_p$ in **Bloc**, and $\mathbf{f}^*$ naturally induces a functor $\mathbf{f}_p^*: \mathbf{BEns}_1(\mathbf{Y}_p) \to \mathbf{BEns}_1(\mathbf{X}_{\mathcal{P}(\mathbf{f})(p)})$.

The notions of an **X**-function, an **X**-functor, etc., can and should be generalized slightly. Let us begin with a generalization of the notion of an **X**-function. An **f**-*function* from a **Y**-set $\mathcal{V}$ to an **X**-set $\mathcal{U}$ is a function $\not{f}$ from the underlying set of $\mathcal{V}$ to that of $\mathcal{U}$ satisfying the following conditions:

(2.1) $\mathcal{P}(\mathbf{f})([\![x = y]\!]_\mathbf{Y}) \leq [\![\not{f}(x) = \not{f}(y)]\!]_\mathbf{X}$
(2.2) $[\![\not{f}(x) = \not{f}(x)]\!]_\mathbf{X} \leq \mathcal{P}(\mathbf{f})([\![x = x]\!]_\mathbf{Y})$

for all $x, y \in \mathcal{V}$. By substituting x for y in (2.1) and taking (2.2) into account, we have

(2.3) $[\![\not{f}(x) = \not{f}(x)]\!]_\mathbf{X} = \mathcal{P}(\mathbf{f})([\![x = x]\!]_\mathbf{Y})$

for all $x \in \mathcal{V}$.

Each **f**-function $\not{f}$ from a small$_1$ **Y**-set $\mathcal{V}$ to a small$_1$ **X**-set $\mathcal{U}$ naturally gives rise to an **X**-function $\not{f}^*: \mathbf{f}^*(\mathcal{V}) \to \mathcal{U}$, for which it is easy to see the following result.

Proposition 2.1. The above assignment $\not{f} \mapsto \not{f}^*$ yields a bijective correspondence between the **f**-functions from $\mathcal{V}$ to $\mathcal{U}$ and the **X**-functions from $\mathbf{f}^*(\mathcal{V})$ to $\mathcal{U}$. ■

Now we introduce a generalization of the notion of an **X**-functor. An **f**-*functor* from a **Y**-category $\mathcal{D}$ to an **X**-category $\mathcal{C}$ is a functor $\mathcal{F}$ from the underlying category of $\mathcal{D}$ to that of $\mathcal{C}$ satisfying the following condition:

(2.4) The restriction of $\mathcal{F}$ to Mor $\mathcal{D}$ is an **f**-function.

Proposition 2.2. The restriction of an **f**-functor $\mathcal{F}: \mathcal{D} \to \mathcal{C}$ to Ob $\mathcal{D}$ is also an **f**-function.

Proof. Proceed as in the proof of Proposition 1.2. ■

For each $p \in \mathcal{P}(\mathbf{Y})$, an **f**-functor $\mathcal{F}: \mathcal{D} \to \mathcal{C}$ naturally gives rise to a functor $\mathcal{F}[p]: \mathcal{D}[p] \to \mathcal{C}[p]$ and an $\mathbf{f}_p$-functor $\mathcal{F}\lceil p: \mathcal{D}\lceil p \to \mathcal{C}\lceil \mathcal{P}(\mathbf{f})(p)$. An **f**-functor from the opposite **Y**-category $\mathcal{D}^{\mathrm{op}}$ of a **Y**-category to an **X**-category $\mathcal{C}$ is called a *contravariant* **f**-*functor* from $\mathcal{D}$ to $\mathcal{C}$.

Let $\mathcal{C}$ be a small$_1$ **X**-category and $\mathcal{D}$ a small$_1$ **Y**-category. Recall that the geometric morphism $(\mathbf{f}_*, \mathbf{f}^*): \mathbf{BEns}_1(\mathbf{X}) \to \mathbf{BEns}_1(\mathbf{Y})$ is open, for which the reader is referred to Nishimura (1993, Theorem 2.13) and MacLane and Moerdijk (1992, Chapter IX, §7, Proposition 2). Since the notion of an **X**-category is a first-order structure in the topos $\mathbf{BEns}_1(\mathbf{X})$ in the sense of MacLane and Moerdijk (1992, Chapter X, §1), the functor $\mathbf{f}^*$: $\mathbf{BEns}_1(\mathbf{Y}) \to \mathbf{BEns}_1(\mathbf{X})$ naturally induces a functor $\mathbf{f}^*_{\mathbf{BCat}}$: $\mathbf{BCat(Y)} \to \mathbf{BCat(X)}$, where $\mathbf{BCat(X)}$ and $\mathbf{BCat(Y)}$ denote the category of small_1 **X**-categories and **X**-functors and that of small_1 **Y**-categories and **Y**-functors, respectively. Each **f**-functor $\mathcal{F}$: $\mathcal{D} \to \mathcal{C}$ naturally gives rise to an **X**-functor $\mathcal{F}^*$: $\mathbf{f}^*_{\mathbf{BCat}}(\mathcal{D}) \to \mathcal{C}$, for which it is easy to see the following result.

Proposition 2.3. The assignment to each **f**-functor $\mathcal{F}$: $\mathcal{D} \to \mathcal{C}$ of $\mathcal{F}^*$ is a bijective correspondence between the **f**-functors from $\mathcal{D}$ to $\mathcal{C}$ and the **X**-functors from $\mathbf{f}^*_{\mathbf{BCat}}(\mathcal{D})$ to $\mathcal{C}$. ■

Such a notion as that of preserving **X**-finite **X**-limits can be generalized easily from **X**-functors to **f**-functors, and we can say, by way of example, that an **f**-functor $\mathcal{F}$: $\mathcal{D} \to \mathcal{C}$ maps **Y**-finite **Y**-limits to **X**-finite **X**-limits. As in Proposition 1.7, we have the following result.

Proposition 2.4. An **f**-functor $\mathcal{F}$: $\mathcal{D} \to \mathcal{C}$ maps **Y**-finite **Y**-limits (**Y**-finite **Y**-colimits, resp.) to **X**-finite **X**-limits (**X**-finite **X**-colimits, resp.) iff the functor $\mathcal{F}[p]$: $\mathcal{D}[p] \to \mathcal{C}[p]$ preserves finite limits (finite colimits, resp.) for all $p \in \mathcal{P}(\mathbf{Y})$. ■

Let $\mathcal{F}$: $\mathcal{D} \to \mathcal{C}$ be an **f**-functor from a **Y**-finitely **Y**-complete **Y**-category $\mathcal{D}$ to a **X**-finitely **X**-complete **X**-category $\mathcal{C}$ mapping **Y**-finite limits to **X**-finite **X**-limits. Then $\mathcal{F}$ naturally gives rise to **f**-functors $\mathcal{F}_{\mathcal{BG}rp}$: $\mathcal{BG}rp(\mathcal{D}) \to \mathcal{BG}rp(\mathcal{C})$ and $\mathcal{F}_{\mathcal{BR}ng}$: $\mathcal{BR}ng(\mathcal{D}) \to \mathcal{BR}ng(\mathcal{C})$.

Example 2.5. The assignments

$$(p, \mathcal{U}) \in \text{Ob } \mathcal{BE}ns_i(Y) \mapsto (\mathcal{P}(\mathbf{f})(p), \mathbf{f}_p^*(\mathcal{U})) \in \text{Ob } \mathcal{BE}ns_i(\mathbf{X})$$

$$(p, f) \in \text{Mor } \mathcal{BE}ns_i(Y) \mapsto (\mathcal{P}(\mathbf{f})(p), \mathbf{f}_p^*(f)) \in \text{Mor } \mathcal{BE}ns_i(X)$$

constitute an **f**-functor $\mathbf{f}^*_{\mathcal{BE}ns_i}$: $\mathcal{BE}ns_i(Y) \to \mathcal{BE}ns_i(X)$ $(i = 0, 1)$. ■

Now we deal with a generalization of the notion of a natural **X**-transformation. Given two **f**-functors $\mathcal{F}$, $\mathcal{G}$ from a **Y**-category $\mathcal{D}$ to an **X**-category $\mathcal{C}$, a *natural* **f**-*transformation* from the **f**-functor $\mathcal{F}$ to the **f**-functor $\mathcal{G}$ is a natural transformation σ from the functor $\mathcal{F}$ to the functor $\mathcal{G}$ satisfying the following condition:

(2.5) The assignment $x \in \text{Ob } \mathcal{D} \mapsto \sigma_x \in \text{Mor } \mathcal{C}$ is an **f**-function.

If σ_x is an isomorphism for each $x \in \text{Ob } \mathcal{D}$, then σ is called a *natural* **f**-*isomorphism*.

The following theorem is a kind of Kan theorem. For the general theory of Kan constructions, the reader is referred to MacLane (1971, Chapter X).

Theorem 2.6. Let $\mathcal{F}$ be a contravariant **f**-functor from a small$_1$ **Y**-category $\mathcal{D}$ to a small$_1$-**X**-cocomplete **X**-category $\mathcal{E}$. Then there is, up to natural **f**-isomorphisms, a unique **f**-functor $\mathcal{G}$: $\mathcal{BF}unct_{\mathbf{Y}}(\mathcal{D}, \mathcal{BE}ns_1(\mathbf{Y})) \to \mathcal{E}$ mapping small$_1$ **Y**-colimits to small$_1$ **X**-colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{BF}unct_{\mathbf{Y}}(\mathcal{D}, \mathcal{BE}ns_1(\mathbf{Y})) & \xrightarrow{\mathcal{G}} & \mathcal{E} \\ \uparrow \mathbf{y} & \nearrow_{\mathcal{F}} & \\ \mathcal{D} & & \end{array}$$

Proof. Booleanize the argument of MacLane and Moerdijk (1992, Chapter I, §5). ■

Theorem 2.7. Let $\mathcal{F}$ be an **f**-functor from a small$_1$ **Y**-category $\mathcal{D}$ to a small$_1$ **X**-category $\mathcal{C}$. Then there is, up to natural **f**-isomorphisms, a unique **f**-functor

$$\mathcal{F}: \mathcal{BF}unct_{\mathbf{Y}}(\mathcal{D}, \mathcal{BE}ns_1(\mathbf{Y})) \to \mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X}))$$

mapping small$_1$ **Y**-colimits to small$_1$ **X**-colimits and making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{BF}unct_{\mathbf{Y}}(\mathcal{D}, \mathcal{BE}ns_1(\mathbf{Y})) & \xrightarrow{\mathcal{F}} & \mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X})) \\ \uparrow \mathbf{y} & & \uparrow \mathbf{y} \\ \mathcal{D} & \xrightarrow[\mathcal{F}]{} & \mathcal{C} \end{array}$$

Proof. Take $\mathcal{BF}unct_{\mathbf{X}}(\mathcal{C}, \mathcal{BE}ns_1(\mathbf{X}))$ for $\mathcal{E}$ in the above theorem. ■

Theorem 2.8. In the above theorem, if we assume also that $\mathcal{D}$ is **Y**-finitely **Y**-cocomplete and that $\mathcal{F}$ maps **Y**-finite **Y**-colimits to **X**-finite **X**-colimits, then $\tilde{\mathcal{F}}$ maps **Y**-finite **Y**-limits to **X**-finite **X**-limits.

Proof. Booleanize the proof of Theorem 17.1.6(e) of Schubert (1972). ■

Let $\mathcal{R}$ be a total object of $\mathcal{BR}ng$ (**X**), $\mathcal{P}$ a total object of $\mathcal{BR}ng$ (**Y**), and f: $\mathbf{f}^*_{\mathcal{BR}ng}(\mathcal{P}) \to \mathcal{R}$ a total morphism of $\mathcal{BR}ng$ (**X**), which shall be fixed throughout the rest of this section. Since the functor $\mathbf{f}^*_{\mathcal{BE}ns_1}$: $\mathcal{BE}ns_1(\mathbf{Y}) \to \mathcal{BE}ns_1(\mathbf{X})$ maps **Y**-finite **Y**-limits to **X**-finite **X**-limits, it naturally induces an **f**-functor $\mathbf{f}^*_{\mathscr{BR}ng}: \mathscr{BR}ng(\mathbf{Y}) \to \mathscr{BR}ng(\mathbf{X})$. Similarly, the **f**-functor $\mathbf{f}^*_{\mathscr{BR}ng}: \mathscr{BR}ng(\mathbf{Y}) \to \mathscr{BR}ng(\mathbf{X})$ naturally induces an **f**-functor

$$\mathbf{f}^{-1}_{\mathscr{BA}lg}: \mathscr{BA}lg_{\mathscr{P}}(\mathbf{Y}) \to \mathscr{BA}lg_{\mathbf{f}^*_{\mathscr{BR}ng}(\mathscr{P})}(\mathbf{X})$$

The morphism $\not{f}: \mathbf{f}^*_{\mathscr{BR}ng}(\mathscr{P}) \to \mathscr{R}$ makes $\mathscr{R}$ an object of

$$\mathscr{BA}lg_{\mathbf{f}^*_{\mathscr{BR}ng}(\mathscr{P})}(\mathbf{X}),$$

which naturally induces a base extension **f**-functor

$$-\otimes_{\mathbf{f}^*_{\mathscr{BR}ng}(\mathscr{P})}\mathscr{R}: \mathscr{BA}lg_{\mathbf{f}^*_{\mathscr{BR}ng}(\mathscr{P})}(\mathbf{X}) \to \mathscr{BA}lg_{\mathscr{R}}(\mathbf{X})$$

We denote by $\mathbf{f}^*_{\mathscr{BA}lg}$ the composition

$$(-\otimes_{\mathbf{f}^*_{\mathscr{BR}ng}(\mathscr{P})}\mathscr{R}) \circ \mathbf{f}^{-1}_{\mathscr{BA}lg}$$

for which we have the following result.

Proposition 2.9. The **f**-functor $\mathbf{f}^*_{\mathscr{BA}lg}: \mathscr{BA}lg_{\mathscr{P}}(\mathbf{Y}) \to \mathscr{BA}lg_{\mathscr{R}}(\mathbf{X})$ maps **Y**-finite **Y**-colimits to **X**-finite **X**-colimits.

Proof. It is sufficient to note that the **f**-functor $\mathbf{f}^{-1}_{\mathscr{BA}lg}$ maps **Y**-finite **Y**-colimits to **X**-finite **X**-colimits and that the **X**-functor

$$-\times_{\mathbf{f}^*_{\mathscr{BA}ng}(\mathscr{P})}\mathscr{R}$$

preserves **X**-finite **X**-colimits. ■

By Theorem 2.7 there exists, up to natural **f**-isomorphisms, a unique **f**-functor

$$\mathbf{f}^*_{\mathscr{BF}unct}: \mathscr{BF}unct_{\mathscr{P}}(\mathbf{Y}) \to \mathscr{BF}unct_{\mathscr{R}}(\mathbf{X})$$

mapping **Y**-colimits to **X**-colimits and making the following diagram commutative:

$$\begin{array}{ccc}
\mathscr{BF}unct_{\mathscr{P}}(\mathbf{Y}) & \xrightarrow{\mathbf{f}^*_{\mathscr{BF}unct}} & \mathscr{BF}unct_{\mathscr{R}}(\mathbf{X}) \\
\uparrow{\scriptstyle \mathbf{y}} & & \uparrow{\scriptstyle \mathbf{y}} \\
\mathscr{BA}lg_{\mathscr{P}}(\mathbf{Y}) & \xrightarrow[\mathbf{f}^*_{\mathscr{BA}lg}]{} & \mathscr{BA}lg_{\mathscr{R}}(\mathbf{X})
\end{array}$$

Proposition 2.10. The **f**-functor

$$\mathbf{f}^*_{\mathscr{BF}unct}: \mathscr{BF}unct_{\mathscr{P}}(\mathbf{Y}) \to \mathscr{BF}unct_{\mathscr{R}}(\mathbf{X})$$

maps **Y**-finite **Y**-limits.

Proof. This follows from Theorem 2.8 and Poropsition 2.9. ■

Example 2.11. Since the **f**-functor

$$\mathbf{f}^*_{\mathfrak{B}\mathcal{F}unct}: \mathfrak{B}\mathcal{F}unct_{\mathcal{P}}(\mathbf{Y}) \rightarrow \mathfrak{B}\mathcal{F}unct_{\mathcal{R}}(\mathbf{X})$$

maps **Y**-finite **Y**-limits to **X**-finite **X**-limits by Proposition 2.10, it naturally gives rise to **f**-functors

$$\mathbf{f}^*_{\mathfrak{B}\mathcal{G}\mathcal{F}unct} = (\mathbf{f}^*_{\mathfrak{B}\mathcal{F}unct})_{\mathfrak{B}\mathcal{G}_{rp}}: \mathfrak{B}\mathcal{G}\mathcal{F}unct_{\mathcal{P}}(\mathbf{Y}) \rightarrow \mathfrak{B}\mathcal{G}\mathcal{F}unct_{\mathcal{R}}(\mathbf{X})$$

$$\mathbf{f}^*_{\mathfrak{B}\mathcal{R}\mathcal{F}unct} = (\mathbf{f}^*_{\mathfrak{B}\mathcal{F}unct})_{\mathfrak{B}\mathcal{R}_{ng}}: \mathfrak{B}\mathcal{R}\mathcal{F}unct_{\mathcal{P}}(\mathbf{Y}) \rightarrow \mathfrak{B}\mathcal{R}\mathcal{F}unct_{\mathcal{R}}(\mathbf{X}) \quad ■$$

Example 2.12. The **Y**-functors $\mathcal{O}_{\mathbf{Y},\mathcal{P}}$ (**X**-functor $\mathcal{O}_{\mathbf{X},\mathcal{R}}$, resp.) is represented by $\mathcal{P}[X]$ ($\mathcal{R}[X]$, resp.). Since $\mathbf{f}^*_{\mathfrak{B}\mathcal{A}\ell g}(\mathcal{P}[X])$ can naturally be identified with $\mathcal{R}[X]$, $\mathbf{f}^*_{\mathfrak{B}\mathcal{R}\mathcal{F}unct}(\mathcal{O}_{\mathbf{Y},\mathcal{P}})$ can naturally be identified with $\mathcal{O}_{\mathbf{X},\mathcal{R}}$. Therefore the **f**-functor $\mathbf{f}^*_{\mathfrak{B}\mathcal{F}unct}$ naturally gives rise to an **f**-functor

$$\mathbf{f}^*_{\mathfrak{B}\mathcal{L}\mathcal{F}unct}: \mathfrak{B}\mathcal{L}\mathcal{F}unct_{\mathcal{P}}(\mathbf{Y}) \rightarrow \mathfrak{B}\mathcal{L}\mathcal{F}unct_{\mathcal{R}}(\mathbf{X}) \quad ■$$

3. QUANTIZATION

Let us begin this section by introducing a category to be denoted by **BCat**. Its objects are all pairs $(\mathbf{X}, \mathcal{C})$ of a Boolean locale **X** and a small$_1$ **X**-category $\mathcal{C}$. A morphism from $(\mathbf{X}, \mathcal{C})$ to $(\mathbf{Y}, \mathcal{D})$ in **BCat** is a pair $(\mathbf{f}, \mathcal{F})$ of a morphism $\mathbf{f}: \mathbf{X} \rightarrow \mathbf{Y}$ in **Bloc** and an **f**-functor $\mathcal{F}: \mathcal{D} \rightarrow \mathcal{C}$. The composition $(\mathbf{g}, \mathcal{G}) \circ (\mathbf{f}, \mathcal{F})$ of morphisms $(\mathbf{f}, \mathcal{F}): (\mathbf{X}, \mathcal{C}) \rightarrow (\mathbf{Y}, \mathcal{D})$ and $(\mathbf{g}, \mathcal{G}): (\mathbf{Y}, \mathcal{D}) \rightarrow (\mathbf{Z}, \mathcal{E})$ in **BCat** is defined to be $(\mathbf{g} \circ \mathbf{f}, \mathcal{F} \circ \mathcal{G})$. The category **BCat** has small$_0$ coproducts. By way of example, given objects $(\mathbf{X}, \mathcal{C})$ and $(\mathbf{Y}, \mathcal{D})$ in **BCat**, their canonical coproduct in **BCat** is $(\mathbf{X} \oplus \mathbf{Y}, \mathcal{C} \oplus \mathcal{D})$, where:

(3.1) $\mathrm{Ob}\,(\mathcal{C} \oplus \mathcal{D}) = \{(x, y) | x \in \mathrm{Ob}\,\mathcal{C} \text{ and } y \in \mathrm{Ob}\,\mathcal{D}\}$.
(3.2) $[\![(x, y) = (x', y')]\!]_{\mathbf{X}\oplus\mathbf{Y}} = ([\![x = x']\!]_{\mathbf{X}}, [\![y = y']\!]_{\mathbf{Y}})$ for all (x, y), $(x', y') \in \mathrm{Ob}\,(\mathcal{C} \oplus \mathcal{D})$.
(3.3) $\mathrm{Mor}\,(\mathcal{C} \oplus \mathcal{D}) = \{(f, g) | f \in \mathrm{Mor}\,\mathcal{C} \text{ and } g \in \mathrm{Mor}\,\mathcal{D}\}$.
(3.4) $[\![(f, g) = (f', g')]\!]_{\mathbf{X}\oplus\mathbf{Y}} = ([\![f = f']\!]_{\mathbf{X}}, [\![g = g']\!]_{\mathbf{Y}})$ for all (f, g), $(f', g') \in \mathrm{Ob}\,(\mathcal{C} \oplus \mathcal{D})$.
(3.5) $\mathrm{d}_{\mathcal{C}\oplus\mathcal{D}}((f, g)) = (\mathrm{d}_{\mathcal{C}}(f), \mathrm{d}_{\mathcal{D}}(g))$ for all $(f, g) \in \mathrm{Mor}\,(\mathcal{C} \oplus \mathcal{D})$.
(3.6) $\mathrm{r}_{\mathcal{C}\oplus\mathcal{D}}((f, g)) = (\mathrm{r}_{\mathcal{C}}(f), \mathrm{r}_{\mathcal{D}}(g))$ for all $(f, g) \in \mathrm{Mor}\,(\mathcal{C} \oplus \mathcal{D})$.
(3.7) $\mathrm{id}_{\mathcal{C}\oplus\mathcal{D}}((x, y)) = (\mathrm{id}_{\mathcal{C}}(x), \mathrm{id}_{\mathcal{D}}(y))$ for all $(x, y) \in \mathrm{Ob}\,(\mathcal{C} \oplus \mathcal{D})$.
(3.8) $(f', g') \circ_{\mathcal{C}\oplus\mathcal{D}} (f, g) = (f' \circ_{\mathcal{C}} f, g' \circ_{\mathcal{D}} g)$ for all

$$((f', g'), (f, g)) \in \mathrm{Mor}\,(\mathcal{C} \oplus \mathcal{D}) \times_{\mathrm{Ob}\,(\mathcal{C}\oplus\mathcal{D})} \mathrm{Mor}\,(\mathcal{C} \oplus \mathcal{D})$$

It is easy to see that the category **BCat** can be put down as an orthogonal category with respect to it small_0 coproduct diagrams for its orthogonal sum diagrams. The assignments $(\mathbf{X}, \mathcal{C}) \in \text{Ob}\,\mathbf{BCat} \mapsto \mathbf{X} \in \text{Ob}\,\mathbf{BLoc}$ and $(\mathbf{f}, \mathcal{F}) \in \text{Mor}\,\mathbf{BCat} \mapsto \mathbf{f} \in \text{Mor}\,\mathbf{BLoc}$ constitute a functor to be denoted by $\boldsymbol{\theta}_{\mathbf{BLoc}}$.

We now introduce a category to be denoted by **BObj**. Its objects are all triples $(\mathbf{X}, \mathcal{C}, \mathscr{x})$ such that $(\mathbf{X}, \mathcal{C}) \in \text{Ob}\,\mathbf{BCat}$ and $\mathscr{x}$ is a total object of the **X**-category $\mathcal{C}$. Its morphisms from $(\mathbf{X}, \mathcal{C}, \mathscr{x})$ to $(\mathbf{Y}, \mathcal{D}, \mathscr{y})$ in **BObj** are all triples $(\mathbf{f}, \mathcal{F}, \mathscr{f})$ such that $(\mathbf{f}, \mathcal{F})$ is a morphism from $(\mathbf{X}, \mathcal{C})$ to $(\mathbf{Y}, \mathcal{D})$ in **BCat** and $\mathscr{f}$ is a total morphism from $\mathcal{F}(\mathscr{y})$ to $\mathscr{x}$ in the **X**-category $\mathcal{C}$. The composition $(\mathbf{g}, \mathcal{G}, \mathscr{g}) \circ (\mathbf{f}, \mathcal{F}, \mathscr{f})$ of $(\mathbf{f}, \mathcal{F}, \mathscr{f})\colon (\mathbf{X}, \mathcal{C}, \mathscr{x}) \to (\mathbf{Y}, \mathcal{D}, \mathscr{y})$ and $(\mathbf{g}, \mathcal{G}, \mathscr{g})\colon (\mathbf{Y}, \mathcal{D}, \mathscr{y}) \to (\mathbf{Z}, \mathcal{E}, \mathscr{z})$ in **BObj** is defined to be $(\mathbf{g} \circ \mathbf{f}, \mathcal{F} \circ \mathcal{G}, \mathscr{f} \circ \mathcal{F}(\mathscr{g}))$. It is easy to see that the category **BObj** has small_0 coproducts. The category **BObj** can be regarded as an orthogonal category with respect to its small_0 coproduct diagrams for its orthogonal sum diagrams. The assignments

$$(\mathbf{X}, \mathcal{C}, \mathscr{x}) \in \text{Ob}\,\mathbf{BObj} \mapsto (\mathbf{X}, \mathcal{C}) \in \text{Ob}\,\mathbf{BCat}$$
$$(\mathbf{f}, \mathcal{F}, \mathscr{f}) \in \text{Mor}\,\mathbf{BObj} \mapsto (\mathbf{f}, \mathcal{F}) \in \text{Mor}\,\mathbf{BCat}$$

constitute a functor from the category **BObj** to the category **BCat** to be denoted by $\boldsymbol{\theta}_{\mathbf{BCat}}$.

Let $\mathfrak{M}$ be a manual of Boolean locales, which shall be fixed throughout the rest of this section. An *empirical framework over* $\mathfrak{M}$ is a functor Φ from $\mathfrak{M}$ to **BCat** satisfying the following conditions:

(3.9) It maps orthogonal $\mathfrak{M}$-sum diagrams to orthogonal sum diagrams in **BCat**.

(3.10) $\boldsymbol{\theta}_{\mathbf{BLoc}} \circ \Phi$ is the identity functor of $\mathfrak{M}$ into **BLoc**.

Example 3.1. The assignments $\mathbf{X} \in \text{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathcal{B}\mathcal{E}ns_i(\mathbf{X}))$ and $\mathbf{f} \in \text{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^*_{\mathcal{B}\mathcal{E}ns_i})$ constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{B}\mathfrak{E}\mathfrak{n}\mathfrak{s}_i$ $(i = 0, 1)$. ∎

Example 3.2. The assignments $\mathbf{X} \in \text{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathcal{B}\mathcal{R}ng(\mathbf{X}))$ and $\mathbf{f} \in \text{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^*_{\mathcal{B}\mathcal{R}ng})$ constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{B}\mathfrak{R}\mathfrak{n}\mathfrak{g}$. ∎

For an empirical framework Φ over $\mathfrak{M}$, we denote by $\Phi_{\mathcal{C}at}(\cdot)$ the function with the same domain of $\Phi(\cdot)$ such that $\Phi(\mathbf{X}) = (\mathbf{X}, \Phi_{\mathcal{C}at}(\mathbf{X}))$ for each $\mathbf{X} \in \text{Ob}\,\mathfrak{M}$ and $\Phi(\mathbf{f}) = (\mathbf{f}, \Phi_{\mathcal{C}at}(\mathbf{f}))$ for each $\mathbf{f} \in \text{Mor}\,\mathfrak{M}$.

Given an empirical framework Φ over $\mathfrak{M}$, we now introduce a category to be denoted by $\mathbf{EObj}(\Phi)$. Its objects are all functors $\mathfrak{F}$ from $\mathfrak{M}$ to **BObj** abiding by the following conditions:

(3.11) It maps orthogonal $\mathfrak{M}$-sum diagrams in $\mathfrak{M}$ to orthogonal sum diagrams in **BObj**.

(3.12) $\boldsymbol{\theta}_{\mathbf{BCat}} \circ \mathfrak{F} = \Phi$.

Given such a functor $\mathfrak{F}: \mathfrak{M} \to \mathbf{BObj}$, we denote by $\mathfrak{F}_{\mathcal{O}bj}$ the function with the same domain of $\mathfrak{F}$ such that the value of $\mathfrak{F}_{\mathcal{O}bj}(\cdot)$ is the third component of the triple $\mathfrak{F}(\cdot)$.

Given such objects $\mathfrak{F}$, $\mathfrak{G}$ in **EObj** (Φ), the morphisms from $\mathfrak{F}$ to $\mathfrak{G}$ in **EObj** (Φ) are all assignments α to each $\mathbf{X} \in \mathrm{Ob}\,\mathfrak{M}$ of a total morphism $\alpha_{\mathbf{X}}$: $\mathfrak{F}_{\mathcal{O}bj}(\mathbf{X}) \to \mathfrak{G}_{\mathcal{O}bj}(\mathbf{X})$ satisfying the following condition:

(3.13) The diagram

$$\begin{array}{ccc}
\Phi_{\mathcal{C}at}(\mathbf{f})(\mathfrak{F}_{\mathcal{O}bj}(\mathbf{Y})) & \xrightarrow{\mathfrak{F}_{\mathcal{O}bj}(\mathbf{f})} & \mathfrak{F}_{\mathcal{O}bj}(\mathbf{X}) \\
{\scriptstyle \Phi_{\mathcal{C}at}(\mathbf{f})(\alpha_{\mathbf{Y}})}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathbf{X}}} \\
\Phi_{\mathcal{C}at}(\mathbf{f})(\mathfrak{G}_{\mathcal{O}bj}(\mathbf{Y})) & \xrightarrow[\mathfrak{G}_{\mathcal{O}bj}(\mathbf{f})]{} & \mathfrak{G}_{\mathcal{O}bj}(\mathbf{X})
\end{array}$$

is commutative for every $\mathbf{f}: \mathbf{X} \to \mathbf{Y} \in \mathrm{Mor}\,\mathfrak{M}$.

As is expected, the composition $\beta \circ \alpha$ of morphisms $\alpha: \mathfrak{F} \to \mathfrak{G}$ and $\beta: \mathfrak{G} \to \mathfrak{H}$ in **EObj** (Φ) is defined to be the assignment $\mathbf{X} \in \mathrm{Ob}\,\mathfrak{M} \mapsto \beta_{\mathbf{X}} \circ \alpha_{\mathbf{X}}$.

The following example is substantially the principal object of our previous research (Nishimura, 1995b).

Example 3.3. Let i be 0 or 1. The objects of the category **EObj** ($\mathfrak{BEns}_i$) are called empirical small$_i$ sets over $\mathfrak{M}$. The category **EObj** ($\mathfrak{BEns}_i$) is called the category of empirical small$_i$ sets over $\mathfrak{M}$. ■

Example 3.4. The objects of the category **EObj** ($\mathfrak{BRng}$) are called empirical rings over $\mathfrak{M}$. The category is called the category of empirical rings over $\mathfrak{M}$. ■

Example 3.5. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$. The assignments $\mathbf{X} \in \mathrm{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathcal{BA}lg_{\mathfrak{R}(\mathbf{X})})$ and $\mathbf{f} \in \mathrm{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^{\sharp}_{\mathcal{A}lg})$ constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{BAlg}_{\mathfrak{R}}$. The objects of the category **EObj** ($\mathfrak{BAlg}_{\mathfrak{R}}$) are called empirical $\mathfrak{R}$-algebras over $\mathfrak{M}$, and the category is called the category of empirical $\mathfrak{R}$-algebras over $\mathfrak{M}$. ■

Example 3.6. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$. The assignments $\mathbf{X} \in \mathrm{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathcal{BF}unct_{\mathfrak{R}(\mathbf{X})})$ and $\mathbf{f} \in \mathrm{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^{\sharp}_{\mathcal{F}unct, \mathfrak{R}(\mathbf{f})})$ constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{BFunct}_{\mathfrak{R}}$. The objects of the category **EObj** ($\mathfrak{BFunct}_{\mathfrak{R}}$) are called empirical $\mathfrak{R}$-functors over $\mathfrak{M}$, and the category is called the category of empirical $\mathfrak{R}$-functors over $\mathfrak{M}$. ■

Example 3.7. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$. The assignments

$$\mathbf{X} \in \mathrm{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathcal{BGF}unct_{\mathfrak{R}_{\mathcal{O}bj}(\mathbf{X})}(\mathbf{X}))$$

and

$$\mathbf{f} \in \text{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^*_{\mathscr{B}\mathscr{G}\mathscr{F}unct})$$

constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{B}\mathfrak{G}\mathfrak{F}\mathfrak{unct}_{\mathfrak{R}}$. The objects of the category **EObj**($\mathfrak{B}\mathfrak{G}\mathfrak{F}\mathfrak{unct}_{\mathfrak{R}}$) are called empirical $\mathfrak{R}$-group-functors over $\mathfrak{M}$, and the category is called the category of empirical $\mathfrak{R}$-group-functors over $\mathfrak{M}$. ■

Example 3.8. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$. The assignments

$$\mathbf{X} \in \text{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathscr{B}\mathscr{R}\mathscr{F}unct_{\mathfrak{R}_{\mathcal{O}_{\mathscr{G}}(\mathbf{X})}}(\mathbf{X}))$$

and

$$\mathbf{f} \in \text{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^*_{\mathscr{B}\mathscr{R}\mathscr{F}unct})$$

constitute an empirical framework offr $\mathfrak{M}$ to be denoted by $\mathfrak{B}\mathfrak{R}\mathfrak{F}\mathfrak{unct}_{\mathfrak{R}}$. The objects of the category **EObj** ($\mathfrak{B}\mathfrak{R}\mathfrak{F}\mathfrak{unct}_{\mathfrak{R}}$) are called empirical $\mathfrak{R}$-ring-functors over $\mathfrak{M}$, and the category is called the category of empirical $\mathfrak{R}$-ring-functors over $\mathfrak{M}$. ■

Example 3.9. Let $\mathfrak{R}$ be an empirical ring over $\mathfrak{M}$. The assignments

$$\mathbf{X} \in \text{Ob}\,\mathfrak{M} \mapsto (\mathbf{X}, \mathscr{B}\mathscr{L}\mathscr{F}unct_{\mathfrak{R}_{\mathcal{O}_{\mathscr{G}}(\mathbf{X})}}(\mathbf{X}))$$

and

$$\mathbf{f} \in \text{Mor}\,\mathfrak{M} \mapsto (\mathbf{f}, \mathbf{f}^*_{\mathscr{B}\mathscr{L}\mathscr{F}unct})$$

constitute an empirical framework over $\mathfrak{M}$ to be denoted by $\mathfrak{B}\mathfrak{L}\mathfrak{F}\mathfrak{unct}_{\mathfrak{R}}$. The objects of the category **EObj** ($\mathfrak{B}\mathfrak{L}\mathfrak{F}\mathfrak{unct}_{\mathfrak{R}}$) are called empirical $\mathfrak{R}$-Lie-functors over $\mathfrak{M}$, and the category is called the category of empirical $\mathfrak{R}$-Lie-functors over $\mathfrak{M}$. ■

4. APPENDIX

Let **X** be a Boolean locale with $\mathbf{B} = \mathscr{P}(\mathbf{X})$, which shall be fixed throughout this appendix. It is well known that the topos $\mathbf{BEns}_i(\mathbf{X})$ enjoys all of classical mathematics ($i = 0, 1$). The purpose of this appendix is to show that the **X**-topos $\mathscr{B}\mathscr{E}ns$ (**X**) also enjoys all of classical mathematics ($i = 0$, 1), which is the very basis of our Booleanization in Section 1. Here we handle only $\mathscr{B}\mathscr{E}ns_1(\mathbf{X})$, leaving a similar treatment of $\mathscr{B}\mathscr{E}ns_0(\mathbf{X})$ to the reader.

Let us begin this section with a brief review of the construction of the Boolean valued universe $V^{(\mathbf{B})}$ of the Zermelo–Fraenkel set theory ZFC with the axiom of choice. For ZFC the reader is referred to a standard textbook on set theory such as Jech (1978). We define $V_\alpha^{(\mathbf{B})}$ by transfinite induction on the ordinal number α in V as follows:

(4.1) $V_0^{(\mathbf{B})} = \varnothing$.
(4.2) $V_\alpha^{(\mathbf{B})} = \{u \mid u: \mathcal{D}(u) \to \mathbf{B} \text{ and } \mathcal{D}(u) \subset \bigcup_{\beta<\alpha} V_\beta^{(\mathbf{B})}\}$.

Then the Boolean valued universe $V^{(\mathbf{B})}$ of Scott and Solovay is defined as follows:

(4.3) $V^{(\mathbf{B})} = \bigcup_{\alpha \in \mathrm{On}(V)} V_\alpha^{(\mathbf{B})}$

where On(V) is the class of all ordinal numbers in V. The class $V^{(\mathbf{B})}$ can be put down as a Boolean valued model of ZFC by defining $[\![u \in v]\!]$ and $[\![u = v]\!]$ for $u, v \in V^{(\mathbf{B})}$ with simultaneous induction

(4.4) $[\![u \in v]\!] = \sup_{y \in \mathcal{D}(v)}(v(y) \wedge [\![u = y]\!])$
(4.5) $[\![u = v]\!] = \inf_{x \in \mathcal{D}(u)}(u(x) \to [\![x \in v]\!]) \wedge \inf_{y \in \mathcal{D}(v)}(v(y) \to [\![y \in u]\!])$

and by assigning $[\![\varphi]\!] \in \mathbf{B}$ to each formula φ without free variables inductively as follows:

(4.6) $[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$.
(4.7) $[\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$.
(4.8) $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \wedge [\![\varphi_2]\!]$.
(4.9) $[\![\forall x\varphi(x)]\!] = \inf_{u \in V^{(\mathbf{B})}}[\![\varphi(u)]\!]$.
(4.10) $[\![\exists x\varphi(x)]\!] = \sup_{u \in V^{(\mathbf{B})}}[\![\varphi(u)]\!]$.

Every theorem of standard mathematics, no matter what branch it belongs to, can be regarded in principle as a theorem of ZFC. The following far-famed theorem gives a powerful transfer principle from standard mathematics to Boolean mathematics.

Theorem 4.1. If φ is a theorem of ZFC, then so is $[\![\varphi]\!] = 1_{\mathbf{B}}$. ∎

The universe V can be embedded into $V^{(\mathbf{B})}$ by transfinite induction as follows:

(4.11) $\check{y} = \{(\check{x}, 1) \mid x \in y\}$ for $y \in V$.

Proposition 4.2. For $x, y \in V$, we have

(1) $[\![\check{x} \in \check{y}]\!] = \begin{cases} 1_{\mathbf{B}} & \text{if } x \in y \\ 0_{\mathbf{B}} & \text{otherwise} \end{cases}$

(2) $[\![\check{x} = \check{y}]\!] = \begin{cases} 1_{\mathbf{B}} & \text{if } x = y \\ 0_{\mathbf{B}} & \text{otherwise} \end{cases}$

The Boolean universe $V^{(\mathbf{B})}$ can be regarded as a category whose objects are all elements of $V^{(\mathbf{B})}$ and whose morphisms are all functions in $V^{(\mathbf{B})}$. We are going to show that the categories $V^{(\mathbf{B})}$ and $\mathbf{BEns}_1(\mathbf{X})$ are equivalent.

Given $u \in V^{(\mathbf{B})}$, we are going to build its associated small_1 **X**-set $\hat{u}$. Each $p \in \mathbf{B}$ determines an equivalence relation $\equiv_p$ on $V^{(\mathbf{B})}[p, u] = \{v \in V^{(\mathbf{B})} | [\![v \in u]\!] \geq p\}$ as follows:

(4.12) $v \equiv_p w$ iff $[\![v = w]\!] \geq p$.

For each $v \in V^{(\mathbf{B})}[p, u]$ we write $[v]_p$ for the equivalence class of v with respect to the equivalence relation $\equiv_p$. The underlying set of $\hat{u}$ shall consist of $[v]_p$'s for all $p \in \mathbf{B}$ and all $v \in V^{(\mathbf{B})}[p, u]$. This $\hat{u}$ can be looked on as an **X**-set with respect to

(4.13) $[\![[v]_p = [w]_q]\!]^{\hat{u}} = [\![v = w]\!] \wedge p \wedge q$ for any $[v]_p, [w]_q \in \hat{u}$.

Every function $f: u \to v$ in $V^{(\mathbf{B})}$ naturally induces an **X**-function $\hat{f}: \hat{u} \to \hat{v}$ as follows:

(4.14) $\hat{f}([w]_p) = [f(w)]_p$ for any $[w]_p \in \hat{u}$.

Conversely, given a small_1 X-set $\mathcal{U}$, we will construct its associated element $\tilde{\mathcal{U}}$ of $V^{(\mathbf{B})}$. First we define an element $\overline{\mathcal{U}}$ of $V^{(\mathbf{B})}$ as follows:

(4.15) $\overline{\mathcal{U}} = \{(\check{x}, [\![x = x]\!]) | x \in \mathcal{U}\}$.

It is not difficult to see that $\sim_{\mathcal{U}} = \{((x, y)^{\vee}, [\![x = y]\!]) | x, y \in \mathcal{U}\}$ is an equivalence relation on $\overline{\mathcal{U}}$ in $V^{(\mathbf{B})}$. The quotient set of $\overline{\mathcal{U}}$ with respect to this equivalence relation in $V^{(\mathbf{B})}$ is denoted by $\tilde{\mathcal{F}}$.

Given a small_1 **X**-function $\mathcal{f}: \mathcal{U} \to \mathcal{V}$, we will construct its associated function $\tilde{\mathcal{f}}$ from $\tilde{\mathcal{U}}$ to $\tilde{\mathcal{V}}$ in $V^{(\mathbf{B})}$. First we define a function $\overline{\mathcal{f}}: \overline{\mathcal{U}} \to \overline{\mathcal{V}}$ in $V^{(\mathbf{B})}$ as follows:

(4.16) $\overline{\mathcal{f}} = \{((x, \mathcal{f}(x))^{\vee}, [\![x = x]\!]) | x \in \mathcal{U}\}$.

Since the function $\overline{\mathcal{f}}$ respects the equivalence relations $\sim_{\mathcal{U}}$ and $\sim_{\mathcal{V}}$ in $V^{(\mathbf{B})}$, it naturally brings forth a function $\tilde{\mathcal{f}}: \tilde{\mathcal{U}} \to \tilde{\mathcal{V}}$ in $V^{(\mathbf{B})}$.

The above considerations give rise to two functors $\mathbb{F}_V: V^{(\mathbf{B})} \to \mathbf{BEns}_1(\mathbf{X})$ and $\mathbb{G}_V: \mathbf{BEns}_1(\mathbf{X}) \to V^{(\mathbf{B})}$. The first functor assigns $\hat{u}$ to each object u of $V^{(\mathbf{B})}$ and $\hat{f}$ to each function f in $V^{(\mathbf{B})}$. The second functor assigns $\tilde{\mathcal{U}}$ to each small_1 **X**-set $\mathcal{U}$ and $\tilde{\mathcal{f}}$ to each **X**-function $\mathcal{f}$ of small_1 **X**-sets. Now it is easy to see the following result.

Theorem 4.3. The categories $V^{(\mathbf{B})}$ and $\mathbf{BEns}_1(\mathbf{X})$ are equivalent.

Proof. It is not difficult to see that the functor $\mathbb{F}_V$ is an equivalence of categories with a quasi-inverse $\mathbb{G}_V$. The details are entrusted to the dexterous reader. ■

By admitting sets of partial existence and using the existential predicate E, we are forced to transform ZFC into ZFC_E. For the metaphysical background of sets of partial existence, the reader is referred to Scott (1979). The formal system ZFC_E can be obtained from Takeuti and Titani's (1981) ZF_I simply by changing the underlying logic from intuitionistic to classical and adding an adequate formulation of the axiom of choice. Due modifications in the construction of $V^{(\mathbf{B})}$ will give rise to a model $V^{\{\mathbf{B}\}}$ of ZFC_E. As in the construction of $V^{(\mathbf{B})}$, we define $V_\alpha^{\{\mathbf{B}\}}$ by transfinite induction on ordinal number α in V as follows:

(4.17) $V_0^{\{\mathbf{B}\}} = \emptyset$.
(4.18) $V_\alpha^{\{\mathbf{B}\}} = \{(p, u) | u: \mathcal{D}(u) \to \mathbf{B}, \mathcal{D}(u) \subset \cup_{\beta<\alpha} V_\beta^{\{\mathbf{B}\}}$ and $u((q, v)) \leq p \wedge q$ for any $(q, v) \in \mathcal{D}(u)\}$.

The Boolean valued universe $V^{\{\mathbf{B}\}}$ is defined as follows:

(4.19) $V^{\{\mathbf{B}\}} = \cup_{\alpha \in \mathrm{On}(V)} V_\alpha^{\{\mathbf{B}\}}$.

The class $V^{\{\mathbf{B}\}}$ can be put down as a Boolean valued model of ZFC_E by defining $[\![(p, u) \in (q, v)]\!]$, $[\![(p, u) = (q, v)]\!]$, and $[\![\mathrm{E}(p, u)]\!]$ for (p, u), $(q, v) \in V^{\{\mathbf{B}\}}$ with simultaneous induction as follows:

(4.20) $[\![(p, u) \in (q, v)]\!] = \sup_{y \in \mathcal{D}(v)}(v(y) \wedge [\![(p, u) = y]\!])$.
(4.21) $[\![(p, u) = (q, v)]\!] = \inf_{x \in \mathcal{D}(u)}(u(x) \to [\![x \in (q, v)]\!]) \wedge \inf_{y \in \mathcal{D}(v)}(v(y) \to [\![y \in (p, u)]\!]) \wedge (p \leftrightarrow q)$.
(4.22) $[\![\mathrm{E}(p, u)]\!] = p$.

We assign $[\![\varphi]\!] \in \mathbf{B}$ to each formula φ without free variables inductively as follows:

(4.23) $[\![\neg\varphi]\!] = \neg[\![\varphi]\!]$.
(4.24) $[\![\varphi_1 \vee \varphi_2]\!] = [\![\varphi_1]\!] \vee [\![\varphi_2]\!]$.
(4.25) $[\![\varphi_1 \wedge \varphi_2]\!] = [\![\varphi_1]\!] \wedge [\![\varphi_2]\!]$.
(4.26) $[\![\forall x\varphi(x)]\!] = \inf_{x \in V^{\{\mathbf{B}\}}}([\![\mathrm{E}x]\!] \to [\![\varphi(x)]\!])$.
(4.27) $[\![\exists x\varphi(x)]\!] = \sup_{x \in V^{\{\mathbf{B}\}}}([\![\mathrm{E}x]\!] \wedge [\![\varphi(x)]\!])$.

The Boolean universe $V^{\{\mathbf{B}\}}$ can be regarded as an **X**-category whose objects are all elements of $V^{\{\mathbf{B}\}}$ and whose morphisms are all functions in $V^{\{\mathbf{B}\}}$. Here, two elements x and y of $V^{\{\mathbf{B}\}}$ are identified provided that $[\![\mathrm{E}x]\!] = [\![\mathrm{E}y]\!] = [\![x = y]\!]$, and a function means an entity f in $V^{\{\mathbf{B}\}}$ such that $[\![f: x \to y]\!] = [\![\mathrm{E}f]\!] = [\![\mathrm{E}x]\!] = [\![\mathrm{E}y]\!]$ for some elements $x, y \in V^{\{\mathbf{B}\}}$. A discussion similar to that leading to Theorem 4.3 gives the following result.

Theorem 4.4. The **X**-categories $V^{\{\mathbf{B}\}}$ and $\mathcal{BE}ns_1(\mathbf{X})$ are **X**-equivalent. ■

Its technical details are safely left to the sagacious reader.

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